ON THE GENERAL EQUATIONS OF INERTIAL NAVIGATION

(OB OBSHCHIKH URAVNENIIAKH INERTSIAL'NOI NAVIGATSII)

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Derived and investigated are the equations of ideal (unperturbed) and perturbed operation of a generalized system of inertial navigation which determines the coordinates of a moving object from the measurements of three linear accelerometers and three angular velocity meters measuring absolute angular velocity. The instruments are located along the axes of an orthogonal trihedron, the origin of which coincides with a certain point in the moving object.

In contrast to [1 and 2], the orientation of the trihedron axes is arbitrary.

The equations of ideal operation take into account the non-central nature of the Earth's gravitational field. The equations of perturbed operation include instrument error effects.

Error equations are derived and investigated not only for small but also for the finite values of the variables of the perturbed operating conditions.

The particular cases are the equations and the results of investigation of the particular control systems considered in [1 to 7].

1. Let us introduce the right-handed orthogonal systems of coordinates $O_2\xi\eta\zeta$ and $O_1\xi\eta\zeta$. The coordinate system $O_2\xi\eta\zeta$ is an inertial system for which Newton's laws of dynamics are valid by definition. The choice of the location of point O_2 and orientation of the axes $\xi\eta\zeta$ are not subject to other conditions. The origin of the system $O_1\xi\eta\zeta$ coincides with the Earth's center. The orientation of the axes of the $O_1\xi\eta\zeta$ system is invariant with respect to the inertial system. Without loss of generality, their orientation can be considered identical.

Let us introduce also the systems of coordinates $O\xi\eta\zeta$ and $Oxyz$ with the origin at a certain point of the moving object (not necessarily at the center of mass). The orientation of the $O\xi\eta\zeta$ system of coordinates coincides with the corresponding orientation of the $O_2\xi\eta\zeta$, $O_1\xi\eta\zeta$ systems of coordinates. The orientation of the $Oxyz$ system is arbitrary.

Finally, let us introduce the system of coordinates O_1xyz , the origin of which is at the Earth's center, while the orientation of the axes is identical to those of the trihedron $Oxyz$.

The problem which is to be solved by the inertial system is the determination of the Cartesian coordinates ξ, η, ζ of the point O in the object within the coordinate system $O_1\xi\eta\zeta$, and the parameters defining the orientation of the object relative to the $\xi\eta\zeta$ axes.

Let along the edges of the $Oxyz$ trihedron, there be installed three accelerometers and three sensors of the projection of the absolute angular velocity of the $Oxyz$ trihedron upon its axes.

We will denote the accelerometer measurements by n_x , n_y , n_z , the projections of the absolute angular velocity measurements by m_x , m_y , m_z , and introduce the vectors

$$\mathbf{n} = n_x\,\mathbf{x} + n_y\,\mathbf{y} + n_z\,\mathbf{z}, \qquad m = m_x\,\mathbf{x} + m_y\,\mathbf{y} + m_z\,\mathbf{z} = \boldsymbol{\omega} \quad (1.1)$$

where $\boldsymbol{\omega}$ is the absolute angular velocity of the trihedron $Oxyz$, and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are the unit vectors of the corresponding axes.

Let us define $\mathbf{n}$, assuming unit accelerometer point masses located at the point O .

If $\mathbf{r}_{02}$ is the radius vector of the point O in an inertial system of coordinates, then the condition for relative equilibrium of the sensing mass of the accelerometer is of the form

$$d^2\mathbf{r}_{02}/dt^2 = \mathbf{F}(\mathbf{r}_{02}) + \mathbf{f} \quad (1.2)$$

where $\mathbf{F}(\mathbf{r}_{02})$ is the geometric sum of the sensing mass attraction forces due to the combined multitude of the celestial bodies, while $\mathbf{f}$ is the sum of the forces acting on the sensing masses through their suspensions.

In an accelerometer the sensing mass is usually suspended elastically [1 and 2]. The magnitude of the elastic deformation of the suspension is proportional to the force $\mathbf{f}$ and is the output of the accelerometer*. Therefor , assuming the proportionality coefficient unity, we have

$$\mathbf{n} = d^2\mathbf{r}_{02}/dt^2 - \mathbf{F}(\mathbf{r}_{02}) \quad (1.3)$$

Note that the differentiation in Formula (1.3) is carried out in the system of coordinates $O_2\xi\eta\zeta$.

If $\mathbf{r}$ denotes the radius vector of the point O relative to O_1 , and $\mathbf{r}_{01}$ is the radius vector of the point O_1 relative to O_2, then by taking into account that

$$\mathbf{r}_{02} = \mathbf{r} + \mathbf{r}_{01}, \qquad d^2\mathbf{r}_{01}/dt^2 = \mathbf{F}(\mathbf{r}_{01}) \quad (1.4)$$

* Here the suspension is assumed to be a spring while in some practical accelerometers the elastic forces may also be different, e.g. electromagnetic.

and by introducing the special designation $\mathbf{g}(\mathbf{r})$ for the Earth attraction force on a unit mass located at O , we get

$$\mathbf{n} = d^2\mathbf{r}/dt^2 - \mathbf{g}(\mathbf{r}) + \mathbf{F}_1(0) - \mathbf{F}_1(\mathbf{r}) \tag{1.5}$$

In (1.5) the $\mathbf{F}_1(\mathbf{r})$ is determined now by the attraction field at the point O of all the celestial bodies with the exception of the Earth.

Let us consider, for the time being, that the object motion (point O) occurs at a small distance above the Earth's surface (compared with its radius, for example). Then the difference

$$\mathbf{F}_1(0) - \mathbf{F}_1(\mathbf{r}) \tag{1.6}$$

of the attraction forces at points O and O_1 becomes negligibly small even for the nearest celestial bodies including the Moon and the Sun. Therefore, with great accuracy

$$\mathbf{n} = d^2\mathbf{r} / dt^2 - \mathbf{g}(\mathbf{r}) \tag{1.7}$$

Since the system of coordinates $O_1\xi\eta\zeta$ translates relative to $O_2\xi\eta\zeta$, the differentiation in the formula (1.7) can be considered carried out in the $O_1\xi\eta\zeta$ system .

2. Let us construct the equations for ideal operation, i.e. the equations for unperturbed functioning of the inertial system.

Projecting on the O_1xyz axes

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}^{\cdot} + \boldsymbol{\omega} \times \mathbf{r}, \qquad \frac{d^2\mathbf{r}}{dt^2} = \left(\frac{d\mathbf{r}}{dt}\right)^{\cdot} + \boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt}^{\cdot} \tag{2.1}$$

where the local derivatives denoted by dots are with respect to the O_1xyz system of coordinates. With the aid of (2.1) from (1.7), and by replacing $\boldsymbol{\omega}$ by $\mathbf{m}$ in accordance with (1.1), we get by integrating in the O_1xyz system

$$\frac{d\mathbf{r}}{dt} = \int_0^t \left[\mathbf{n} - \mathbf{m} \times \frac{d\mathbf{r}}{dt} + \mathbf{g}(\mathbf{r})\right] dt + \frac{d\mathbf{r}^\circ}{dt}, \qquad \mathbf{r} = \int_0^t \left(\frac{d\mathbf{r}}{dt} - \mathbf{m} \times \mathbf{r}\right) dt + \mathbf{r}^\circ \tag{2.2}$$

With the aid of the computers the relationshps (2.2) permit determination of the Cartesian coordinates x, y, z of the point O in the O_1xyz coordinate system for the quantities n_x, n_y, n_z ; m_x, m_y, m_z and the initial values $\mathbf{r}^\circ$, $\frac{d\mathbf{r}^\circ}{dt}$, provided, of course, that $g_x(x,y,z)$, $g_y(x,y,z)$ and $g_z(x,y,z)$ are known, which will be the case if, for example, the Earth's gravitational field is central (more correctly, spherical).

In order to pass to the Cartesian coordinates ξ, η, ζ of the point O in the system $O_1\xi\eta\zeta$, it is necessary to model the equation

$$\boldsymbol{\xi} = \int_0^t (\boldsymbol{\xi} \times \mathbf{m})\, dt + \boldsymbol{\xi}^0 \qquad (\xi\eta\zeta) \tag{2.3}$$

along with (2.2) which permits, for m_x, m_y, m_z and the mutual disposition of the systems of coordinates $O_1\xi\eta\zeta$ and O_1xyz at the initial time, the determination of the direction cosines between the unit vectors $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ and $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ and Equations

$$\xi = \boldsymbol{\xi}\cdot\mathbf{r} \qquad (\xi\eta\zeta) \tag{2.4}$$

which effect the passage from the coordinates x, y, z to the coordinates ξ, η, ζ In (2.3) and (2.4) and the sequel, the symbol $(\xi\eta\zeta)$ placed near the formula denotes cyclic interchange of the variables and the indices.

It is obvious that the orientation of an object in space is given by its position relative to the trihedron $Oxyz$ which requires an additional measurement either of the position of the trihedron $Oxyz$ relative to the object, or the angular velocity of rotation of the set $Oxyz$ relative to the object and the solution of the equations similar to (2.3).

If the Earth's gravitational field is assumed spherical, i.e.

$$\mathbf{g}(\mathbf{r}) = -\frac{\mathbf{r}}{r}g(r) \tag{2.5}$$

then Equations (2.2) and (2.3) are solved independently. In the opposite case, the gravitational field can be given only in the Earth's coordinate system or in a coordinate system wherein the motion of the Earth is known, for example, in the $O_1\xi\eta\zeta$ coordinate system. In this case the given functions will be $g_\xi(\xi, \eta, \zeta, t)$, $g_\eta(\xi, \eta, \zeta, t)$, and $g_\zeta(\xi, \eta, \zeta, t)$. In order to determine g_x, g_y and g_z it is first required to obtain the solution of Equation (2.3), and then solve simultaneously (2.2) and (2.4).

It is understood that knowing the Cartesian coordinates ξ, η, ζ it is possible to obtain any other coordinates, in the general case curvilinear and nonorthogonal coordinates $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, which may be dependent on ξ, η, ζ and time t if the surfaces of equal values of the coordinates $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ will change their position relative to the trihedron $O_1\xi\eta\zeta$ with time. The latter will take place, for instance, if the coordinates $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ are rigidly attached to the Earth.

In order to pass to the coordinates $\varkappa_1$, $\varkappa_2$, $\varkappa_3$, it is obviously necessary to specify three relationships of the type

$$\Phi_i(\xi, \eta, \zeta; \varkappa_1, \varkappa_2, \varkappa_3, t) = 0 \qquad (i = 1, 2, 3) \tag{2.6}$$

which in the whole performance region of the inertial system must satisfy the usual conditions of unique correspondence of the ξ, η, ζ and $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ coordinates, i.e. the determinants of the Jacobi functions Φ_1, Φ_2, Φ_3 of ξ, η, ζ and $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ must be nonzero.

It is worth to note that the construction of the equations for ideal performance in their integral form (the form in which they are simulated by the computer of the inertial navigation system [2]) is not entirely unique. They may be constructed in several equivalent but different forms.

For example, in place of Equation (2.2) one may take the equivalent equation

$$\mathbf{r} = \int_0^t \left[\int_0^t (\mathbf{n} - \mathbf{m}\times(\mathbf{r}^{\cdot} + \mathbf{m}\times\mathbf{r}) + \mathbf{g}(\mathbf{r}))\,dt + (\mathbf{r}^{\cdot})^\circ + \boldsymbol{\omega}^\circ\times\mathbf{r}^\circ\right] dt + \mathbf{r}^\circ \tag{2.7}$$

Equations (2.2) and (2.7) presuppose integration along those axes in $Oxyz$ (O_1xyz), where n_x, n_y, n_z and m_x, m_y, m_z are measured, and obtaining the Cartesian coordinates x, y, z.

Another construction is essentially different. Utilizing (2.3), $\mathbf{n}$ can be projected on the axes of the set $O_1\xi\eta\zeta$ with integration carried out along the axes of this set and then obtaining the Cartesian coordinates ξ, η, ζ. One can project $\mathbf{n}$ along the axes of any other orthogonal system $O_1\xi'\eta'\zeta'$ (with the origin at the geocenter O_1) the motion of which is known relative to $O_1\xi\eta\zeta$, the axes of such system may be, for example, rigidly attached to the Earth. Finally, one may project $\mathbf{n}$ along the normals to the coordinate surfaces $\varkappa_i$ = const, carry out the integration along these normals and obtain as the result of integration the coordinates $\varkappa_1$, $\varkappa_2$, $\varkappa_3$.

A possible intermediate method for constructing the equations for ideal performance is to carry out the first integration along one set of directions (not necessarily coinciding with the directions of the $\mathbf{n}$ components measurements), and the second integration along another set of directions.

Equations (2.3) can also be expressed in other forms, for example, by introduction of Euler angles, the Olding Rodrigues parameters or the Cayley - Klein parameters [8 and 9].

All variations for constructing the equations of ideal performance differ insignificantly in the number of required computer operations if one considers the complete equations for ideal performance, an arbitrary orientation of the trihedron $Oxyz$, and an arbitrary motion of the object. The possibility of simplifying the equations for ideal performance is dependent on the choice and the maintenance of a special orientation of the $Oxyz$ trihedron, on the imposition on the motion of the object of definite limitations, and on the neglecting of certain terms in the equations for ideal performance, the inclusion of which leads to the accuracy beyond that provided by the inertial system.

The coordinate set $Oxyz$ can be rigidly attached to the object. Then the complete system of equations for ideal performance is utilized. The solution of the system (2.3) at the same time also determines the orientation of the object in space.

The coordinate set $Oxyz$ may be fixed in space [2], for example, its axes may be parallel to $\xi\eta\zeta$. Then the integration yields immediately the coordinates ξ, η, ζ and Equations (2.3) fall out. Such orientation of the set $Oxyz$ can be ensured by a stabilized platform.

If the axes of the set $Oxyz$ are parallel to the axes ξ', η', ζ', the orientation of which in the system $O_1\xi\eta\zeta$ is a known function of time, then the Cartesian coordinates ξ', η', ζ' are obtained by direct integration. At the same time the position of the set $Oxyz$ relative to the axes of $O\xi\eta\zeta$ must repeat in time the given orientation of the system $O_1\xi'\eta'\zeta'$ relative to $O_1\xi\eta\zeta$.

The coordinate set $Oxyz$ can be oriented also with regard to the coordinates $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ of the point O of the object as determined by the inertial system itself. For example, one of its surfaces can be a tangent surface to the area of constant value of some coordinate, for instance, $\varkappa_1$, then one axis of the set $Oxyz$, such as Ox, is normal to this surface. In this case the integration along the axis Ox will yield immediately the coordinate $\varkappa_1$. If the coordinates $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ are orthogonal, then it is possible to locate the axes Oy and Oz along the normals to the surfaces $\varkappa_2$ = const and $\varkappa_3$ = const and determine $\varkappa_2$, $\varkappa_3$ by integration. In the case of nonorthogonal set of coordinates it is possible to pass to the nonorthogonal trihedron $Oxyz$.

Examples of the orientation of an orthogonal set $Oxyz$ which account for the present position of the object are: the realization of an tracking Darboux trihedron on a sphere of radius r, concentric with the Earth, if the coordinates $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ are spherical coordinates (geocentric or geodesic), or a tracking trihedron on the surface h = const if the defining coordinates are the latitude, longitude, and height h above sea level.

The simplest example of expedient transformation to a nonorthogonal trihedron is the case when the defining coordinates are: the distance r from the center of the Earth O_1 and the angles $\varkappa_1$ and $\varkappa_2$ between $\mathbf{r}$ and the axes $O_1\xi$ and $O_1\eta$. In this case, having located the axis Oz along $\mathbf{r}$, the axes Ox and Oy should be located in the planes $OO_1\xi$ and $OO_1\eta$. It is clear that in this case the axes Ox, Oy and Oz are normal to the surfaces $\varkappa_1$ = const, $\varkappa_2$ = const and r = const, and the angle between the axes Ox and Oy is not a right angle and depends upon the coordinates $\varkappa_1$, $\varkappa_2$.

If the object moves along the surface $\varkappa_1$ = const, then the equations for ideal performance are simplified by the fact that the part related to the definition of $\varkappa_1$ is eliminated so that no accelerometer along the normal to the surface $\varkappa_1$ = const should be needed. Analogous simplifications are possible when the surface is not a coordinate but is given by the relationship $\sigma_1(\varkappa_1, \varkappa_2, \varkappa_3) = 0$. Example of such simplified systems with two accelerometers (or equivalent systems with two accelerometers) are the control systems presented in [1 to 5].

Finally, if the object moves along the line $\sigma_1(\varkappa_1, \varkappa_2, \varkappa_3) = 0$, $\sigma_2(\varkappa_1, \varkappa_2, \varkappa_3) = 0$, then the second accelerometer may be removed from the system with the remaining one being oriented along the tangent of this line.

Of course, the enumerated simplifications are conditional on the imposition of constraints upon the motion of the object. In particular, however, the elimination of one or even two accelerometers is possible in the absence of such constraints if one or two coordinates are determined not by the inertial system but by means of other sources of information. For example, the distance to the Earth's center may be determined by means of a radio altimeter [2].

We will note also that along with the omission of some negligible parts in the complete equations for ideal performance, the simplification of these equations may also be effected by forming part of the terms not as functions of the present performance of the object's motion, but as functions of its programed values, i.e. as functions of time.

The equations for ideal performance of the inertial system considered above, were formed with the assumption that the motion occurs at such proximity to the Earth's surface that in (1.5) the difference (1.6) can be neglected. It is easy to see that this limiting assumption need not be made.

Let there be n celestial bodies, the attraction of which should be evaluated, taking the difference (1.6) into account.

We will denote by $\mathbf{r}_i$ the radius-vector of the center of mass for the i-th celestial body relative to the Earth's center of mass O_1. Then the radius-vector $\mathbf{r}_i^{(1)}$ of the point O relative to the center of the i-th body is equal to

$$\mathbf{r}_i^{(1)} = \mathbf{r} - \mathbf{r}_i \tag{2.8}$$

If one considers that the masses of the considered celestial bodies and their motions relative to the Earth are known (the object, obviously, does not perturb the motion of the Earth or that of the celestial bodies), and that their attraction fields are central, then [10]

$$\mathbf{F}_1(0) - \mathbf{F}_1(\mathbf{r}) = \sum_{i=1}^{n} m_i \gamma \left(\frac{\mathbf{r}_i}{r_i^3} - \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} \right) \tag{2.9}$$

where m_i is the mass of the i-th celestial body, and γ a gravitational constant.

Taking into account (2.9), the first equation in (2.2) is replaced by Equation

$$\frac{d\mathbf{r}}{dt}=\int_0^t\left[\mathbf{n}-\mathbf{m}\times\frac{d\mathbf{r}}{dt}+\mathbf{g}(\mathbf{r})+\sum_{i=1}^{n}m_i\gamma\left(\frac{\mathbf{r}_i}{r_i^3}-\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}\right)\right]dt+\frac{d\mathbf{r}^\circ}{dt} \qquad (2.10)$$

The projections of $\mathbf{r}_i$ on the axes of O_1xyz in (2.16) are determined by the projections on the axes of $O_1\xi\eta\zeta$, which are known as functions of time, and by Formulas (2.3)

The consideration of noncentrality in the fields of attraction of the considered celestial bodies does not introduce principal difficulties in the construction of the equations for ideal performance but does make these equations substantially more unwieldy, since the determination of the attraction fields requires introduction of n systems of coordinates each of which is rigidly attached to the i-th celestial body. In this case, it becomes necessary that the motion of each celestial body about its center of mass with respect to the system of coordinates $O_2\xi\eta\zeta$ (or $O_1\xi\eta\zeta$) be known.

3. Let us derive the error equations, i.e. the equations describing the perturbed operation of the inertial system when the initial conditions of the equations for ideal performance are given inaccurately and the elements of the system have instrument errors. The error equations determine the stability of performance of the inertial system and the dependence of its accuracy upon the magnitude of errors in setting up the initial conditions and upon the instrument errors of the elements.

As the instrument errors we will take the basic ones: $\Delta\mathbf{m}$ and $\Delta\mathbf{n}$ [2]. It can be shown that any other errors can always be reduced to the equivalent basic ones.

From (2.2), assuming now

$$\mathbf{r}'=\mathbf{r}+\delta\mathbf{r},\qquad \mathbf{m}'=\boldsymbol{\omega}+\Delta\mathbf{m},\qquad \mathbf{n}'=\mathbf{n}+\Delta\mathbf{n} \qquad (3.1)$$

we obtain

$$\frac{d\mathbf{r}'}{dt}=\int_0^t[\mathbf{n}'-\mathbf{m}'\times\frac{d\mathbf{r}'}{dt}+\mathbf{g}(\mathbf{r}')]\,dt+\left(\frac{d\mathbf{r}'}{dt}\right)^\circ$$
$$\mathbf{r}'=\int_0^t\left(\frac{d\mathbf{r}'}{dt}-\mathbf{m}'\times\mathbf{r}'\right)dt+(\mathbf{r}')^\circ \qquad (3.2)$$

Subtracting from (3.2) the equations for ideal performance (2.2) and noting that $\mathbf{m}=\boldsymbol{\omega}$ we obtain, using the notation (3.1), the integral error equations

$$\frac{d\delta\mathbf{r}}{dt}=\int_0^t\Big[\Delta\mathbf{n}-\Delta\mathbf{m}\times\frac{d\mathbf{r}}{dt}-(\boldsymbol{\omega}+\Delta\mathbf{m})\times\frac{d\delta\mathbf{r}}{dt}+$$
$$+\mathbf{g}(\mathbf{r}+\delta\mathbf{r})-\mathbf{g}(\mathbf{r})\Big]dt+\frac{d\delta\mathbf{r}^\circ}{dt} \qquad (3.3)$$
$$\delta\mathbf{r}=\int_0^t\left(\frac{d\delta\mathbf{r}}{dt}-\Delta\mathbf{m}\times\mathbf{r}-(\boldsymbol{\omega}+\Delta\mathbf{m})\times\delta\mathbf{r}\right)dt+\delta\mathbf{r}^\circ$$

Differentiating in the same system of coordinates in which the integrals are taken, i.e. in the $Oxyz$ system, we obtain the differential equation

$$\delta\mathbf{r}^{\cdot\cdot} + 2(\boldsymbol{\omega} + \Delta\mathbf{m}) \times \delta\mathbf{r}^{\cdot} + (\boldsymbol{\omega} + \Delta\mathbf{m}) \times [(\boldsymbol{\omega} + \Delta\mathbf{m}) \times \delta\mathbf{r}] + $$
$$+ (\boldsymbol{\omega}^{\cdot} + \Delta\mathbf{m}^{\cdot}) \times \delta\mathbf{r} - \mathbf{g}(\mathbf{r} + \delta\mathbf{r}) + \mathbf{g}(\mathbf{r}) = \Delta\mathbf{n} - 2\Delta\mathbf{m} \times \mathbf{r}^{\cdot} - \Delta\mathbf{m}^{\cdot} \times \mathbf{r} -$$
$$- \Delta\mathbf{m} \times (\boldsymbol{\omega} \times \mathbf{r}) - (\boldsymbol{\omega} + \Delta\mathbf{m}) \times (\Delta\mathbf{m} \times \mathbf{r}) \tag{3.4}$$

with the initial conditions

$$\delta\mathbf{r}(0) = \delta\mathbf{r}^{\circ}$$
$$\delta\mathbf{r}^{\cdot}(0) = (\delta\mathbf{r}^{\cdot})^{\circ} + (\delta\boldsymbol{\omega}^{\circ} - \Delta\mathbf{m}^{\circ}) \times (\mathbf{r}^{\circ} + \delta\mathbf{r}^{\circ}) \tag{3.5}$$

For the given quantities $\boldsymbol{\omega}, \Delta\mathbf{m}, \Delta\mathbf{n}, \mathbf{r}$ projected on the $Oxyz$ axes and the initial values $\delta\mathbf{r}^{\circ}, \delta\boldsymbol{\omega}^{\circ} (\delta\mathbf{r}^{\cdot})^{\circ}$, the equation (3.4) yields the errors in determination of the Cartesian coordinates x,y,z by the inertial system. Equation (3.4) is exact. If in it the products of the projections $\Delta\mathbf{m}$ and $\delta\mathbf{r}$ are neglected, there results Equation

$$\delta\mathbf{r}^{\cdot\cdot} + 2\boldsymbol{\omega} \times \delta\mathbf{r}^{\cdot} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \delta\mathbf{r}) + \boldsymbol{\omega}^{\cdot} \times \delta\mathbf{r} - \mathbf{g}(\mathbf{r} + \delta\mathbf{r}) + \mathbf{g}(\mathbf{r}) =$$
$$= \Delta\mathbf{n} - 2\Delta\mathbf{m} \times \mathbf{r}^{\cdot} - \Delta\mathbf{m}^{\cdot} \times \mathbf{r} - \Delta\mathbf{m} \times (\boldsymbol{\omega} \times \mathbf{r}) - \boldsymbol{\omega} \times (\Delta\mathbf{m} \times \mathbf{r}) \tag{3.6}$$

with the initial conditions

$$\delta\mathbf{r}(0) = \delta\mathbf{r}^{\circ}, \qquad \delta\mathbf{r}^{\cdot}(0) = (\delta\mathbf{r}^{\cdot})^{\circ} + (\delta\boldsymbol{\omega}^{\circ} - \Delta\mathbf{m}^{\circ}) \times \mathbf{r}^{\circ} \tag{3.7}$$

For $\Delta\mathbf{m} = 0$, i.e. when the perturbations are merely due to the accelerometer and the initial conditions errors, Equation (3.4) is equivalent to (3.6).

In (3.6) $\boldsymbol{\omega}, \mathbf{r}, \Delta\mathbf{m}, \delta\mathbf{r}$ are given in the projections on the O_1xyz axes and $\boldsymbol{\omega}$ is the absolute angular velocity of rotation for the O_1xyz trihedron. Therefore (3.6) can also be written as

$$\frac{d^2\delta\mathbf{r}}{dt^2} + \mathbf{g}(\mathbf{r}) - \mathbf{g}(\mathbf{r} + \delta\mathbf{r}) = \Delta\mathbf{n} - \Delta\mathbf{m} \times \frac{d\mathbf{r}}{dt} - \frac{d}{dt}(\Delta\mathbf{m} \times \mathbf{r}) \tag{3.8}$$

where $\delta\mathbf{r}, \mathbf{r}, \Delta\mathbf{m}, \Delta\mathbf{n}$ are now determined by the projections on the axes of the $O_1\xi\eta\zeta$ coordinate set. Note that the homogeneous equation (3.8) corresponds exactly not only to (3.6) but also to (3.4) where the products of the $\Delta\mathbf{m}$ and $\delta\mathbf{r}$ projections are retained.

Let us turn to the second group of equations for the inertial system performance, i.e. to Equations (2.3) and (2.4).

We have from (2.3)

$$\delta\boldsymbol{\xi}^{\cdot} + (\boldsymbol{\omega} + \Delta\mathbf{m}) \times \delta\boldsymbol{\xi} = \boldsymbol{\xi} \times \Delta\mathbf{m} \qquad (\xi\eta\zeta) \tag{3.9}$$

Or, with the same eliminations as in passing from (3.4) to 3.6)

$$\delta\boldsymbol{\xi}^{\cdot} + \boldsymbol{\omega} \times \delta\boldsymbol{\xi} = \boldsymbol{\xi} \times \Delta\mathbf{m} \qquad (\xi\eta\zeta) \tag{3.10}$$

or finally, analogous to (3.8)

$$\frac{d\delta\boldsymbol{\xi}}{dt} = \boldsymbol{\xi} \times \Delta\mathbf{m} \qquad (\xi\eta\zeta) \tag{3.11}$$

where as in (3.8) $\delta\boldsymbol{\xi}$, $\delta\boldsymbol{\eta}$, $\delta\boldsymbol{\zeta}$ and $\Delta\mathbf{m}$ are defined in the projections on the $O_1\xi\eta\zeta$ axes. The initial conditions for Equations (3.9), (3.10) and (3.11) are obvious.

From (2.4) we have

$$\delta\xi = \delta\boldsymbol{\xi}\cdot\mathbf{r} \qquad (\xi\eta\zeta) \tag{3.12}$$

Denoting by $\delta\mathbf{r}_1$ the full error, and by $\delta\mathbf{r}_2$ the error defined by the second group of equations (3.9) and (3.12) or (3.10) and (3.12), we get

$$\delta\mathbf{r}_1 = \delta\mathbf{r} + \delta\mathbf{r}_2 \tag{3.13}$$

where

$$\delta\mathbf{r}_2 = (\delta\boldsymbol{\xi}\cdot\mathbf{r})\,\boldsymbol{\xi} + (\delta\boldsymbol{\eta}\cdot\mathbf{r})\,\boldsymbol{\eta} + (\delta\boldsymbol{\zeta}\cdot\mathbf{r})\boldsymbol{\zeta} \tag{3.14}$$

and $\delta\mathbf{r}$ is the solution of the error equations of the first group (3.4) or (3.6).

In concluding the deviation of the error equations let us pass from the vector equations (3.6), (3.10), (3.14) and (3.13) to scalar ones. If the variation in the noncentricality of the Earth's gravitational field is neglected and only the linear terms retained in the expansion of the difference $\mathbf{g}(\mathbf{r}) - \mathbf{g}(\mathbf{r}+\delta\mathbf{r})$, then

$$\mathbf{g}(\mathbf{r}) - \mathbf{g}(\mathbf{r}+\delta\mathbf{r}) = \delta\frac{\mathbf{r}}{r}\,g(r) = \frac{g(r)}{r^2}\,[r\delta\mathbf{r} - 3\mathbf{r}\,(|\mathbf{r}+\delta\mathbf{r}| - r)] \tag{3.15}$$

Projecting now (3.6) on the axes xyz and taking into account (3.15), we get the first group of error equations in the form

$$\begin{gathered}\delta x^{\cdot\cdot} + \left(\frac{g}{r} - \omega_y^2 - \omega_z^2\right)\delta x + (\omega_x\omega_y - \omega_z^{\cdot})\,\delta y - \\ - 2\omega_z\,\delta y^{\cdot} + (\omega_x\omega_z + \omega_y^{\cdot})\,\delta z + 2\omega_y\delta z^{\cdot} - \frac{3g}{r^2}\,x\,(|\mathbf{r}+\delta\mathbf{r}| - r) = \\ = \Delta n_x - 2\,(\Delta m_y z^{\cdot} - \Delta m_z y^{\cdot}) - (\Delta m^{\cdot}_y z - \Delta m^{\cdot}_z\, y) - \omega_x\,(\Delta m_y y + \Delta m_z\, z) - \\ - \Delta m_x\,(\omega_y\ \ y + \omega_z\, z) + 2x\,(\omega_y\,\Delta m_y + \omega_z\,\Delta m_z) \qquad (xyz)\end{gathered} \tag{3.16}$$

The initial conditions of this group of equations are

$$\begin{gathered}\delta x\,(0) = \delta x^{\circ}, \\ \delta x^{\cdot}\,(0) = (\delta x^{\cdot})^{\circ} + (\delta\omega_y^{\circ} - \Delta m_y^{\circ})\,z^{\circ} - (\delta\omega_z^{\circ} - \Delta m_z^{\circ})y^{\circ} \qquad (xyz)\end{gathered} \tag{3.17}$$

Let us find the equations for the projections on the axes xyz of δx_2, δy_2, δz_2 of the vector $\delta\mathbf{r}_2$ given by the relationships (3.10) and (3.14).

Introducing the table of direction cosines

	x	y	z
ξ	α_{11}	α_{12}	α_{13}
η	α_{21}	α_{22}	α_{23}
ζ	α_{31}	α_{32}	α_{33}

(3.18)

and observing that Equations (2.3) are equivalent to three systems of scalar differential equations [9]

$$\alpha_{i1}^{\cdot} + \omega_y\alpha_{i3} - \omega_z\alpha_{i2} = 0 \qquad (i = 1, 2, 3) \qquad (123,\ xyz) \tag{3.19}$$

we obtain in place of (3.10) (3.20)

$$\delta\alpha_{i1}^{\cdot} + \omega_y \, \delta\alpha_{i3} - \omega_z \delta\alpha_{i2} = \Delta m_z \alpha_{i2} - \Delta m_y \alpha_{i3} \qquad (i = 1, 2, 3) \qquad (123,\ xyz)$$

Introducing the notation θ_x, θ_y and θ_z for the quantities

$$\begin{aligned} \theta_x &= -\alpha_{12}\delta\alpha_{13} - \alpha_{22}\delta\alpha_{23} - \alpha_{32}\delta\alpha_{33} \\ \theta_y &= \alpha_{11}\delta\alpha_{13} + \alpha_{21}\delta\alpha_{23} + \alpha_{31}\delta\alpha_{33} \\ \theta_z &= -\alpha_{11}\,\delta\alpha_{12} - \alpha_{21}\delta\alpha_{22} - \alpha_{31}\delta\alpha_{32} \end{aligned} \qquad (3.21)$$

and neglecting the squares of the variations of direction cosines α_{ij} we get from (3.19) and (3.20) the relationships

$$\theta_x^{\cdot} + \omega_y\theta_z - \omega_z\theta_y = \Delta m_x \qquad (xyz) \qquad (3.22)$$

But θ_x, θ_y and θ_z are, according to (3.21), the components of small rotation $\boldsymbol{\theta}$ along the axes xyz. Therefore, (3.22) can be written in the form of Equation

$$\boldsymbol{\theta}^{\cdot} + \boldsymbol{\omega} \times \boldsymbol{\theta} = \Delta\mathbf{m} \qquad \left(\frac{d\boldsymbol{\theta}}{dt} = \Delta\mathbf{m}\right) \qquad (3.23)$$

with the initial condition $\boldsymbol{\theta}(0) = \boldsymbol{\theta}^{\circ}$ determined by (3.21).

Now it is obvious that $\delta\mathbf{r}_2 = \boldsymbol{\theta} \times \mathbf{r}$, wherefrom

$$\delta x_2 = \theta_y z - \theta_z y \qquad (xyz) \qquad (3.24)$$

The full error, according to (3.13), is

$$\delta x_1 = \delta x + \delta x_2 \qquad (xyz) \qquad (3.25)$$

where δx, δy and δz are defined by (3.16), and δx_2, δy_2 and δz_2 are given by (3.22) and (3.24).

If in computing the difference (3.15) one considers the variation in the noncentrality of the Earth's gravitational field, then it is necessary to retain the quadratic terms in the expansion of the spherical part of the gravitational field since they are of the same order of magnitude as the linear part of the expansion for the correction of nonsphericity (at least for the Earth's field, in any case).

Introducing the notation

$$\mathbf{g}(\mathbf{r}) = -\frac{\mathbf{r}}{r}\, g(r) + \boldsymbol{\varepsilon}(\mathbf{r}) \qquad (3.26)$$

where

$$\boldsymbol{\varepsilon}(\mathbf{r}) = \varepsilon_x(x, y, z)\,\mathbf{x} + \varepsilon_y(x, y, z)\,\mathbf{y} + \varepsilon_z(x, y, z)\,\mathbf{z}$$

is a vector function of $\mathbf{r}$ characterizing the nonsphericity of the attraction field, we get

$$\mathbf{g}(\mathbf{r}) - \mathbf{g}(\mathbf{r} + \delta\mathbf{r}) = -\frac{\delta\mathbf{r}}{r}\, g(r) + \frac{g(r)}{r^2}(\mathbf{r} + \delta\mathbf{r})\left[15\,(|\mathbf{r} + \delta\mathbf{r}| - r) - \frac{12\mathbf{r}\delta\mathbf{r}}{r} - \frac{6\delta\mathbf{r}^2}{r}\right] -$$

$$- (\operatorname{grad} \varepsilon_x \delta\mathbf{r}_1)\,\mathbf{x} - (\operatorname{grad} \varepsilon_y \delta\mathbf{r}_1)\,\mathbf{y} - (\operatorname{grad} \varepsilon_z \delta\mathbf{r}_1)\,\mathbf{z} \qquad (3.27)$$

Since (3.27) contains not only $\delta\mathbf{r}$ but also $\delta\mathbf{r}_1$, then in this case the first group of the error equations is not separated from the second group. Similarly, the equations for ideal performance are not separated in considering the noncentrality of the attraction field.

It is easy to include in the first group of error equations the attraction differences of other (other than Earth) celestial bodies at the Earth's center and at the point O of the present position of the object in accordance with the equations for ideal performance (2.10).

In this case $\delta \mathbf{r}_1 = \delta r_{12}$, and the error equations of both groups are also related.

4. Let us indicate now certain general properties of the error equations for the investigated generalized control system of inertial navigation, and also show how to obtain from them the error equations for particular systems considered in [1 to 5].

The error equations (3.16), (3.22), (3.24) and (3.25) permit a group of transformations determined by the arbitrary rotation of the O_1xyz $(Oxyz)$ trihedron in space. This property of the equations follows from the arbitrary orientation of the coordinate set (trihedron) $Oxyz$ and the arbitrary $\boldsymbol{\omega}$. It can also be proved directly, analogous to the proof in [2]. Therefore the analysis of the error equations for the inertial navigation system for any orientation of the set $Oxyz$ (O_1xyz) can be carried out relative to another set of coordinates (trihedron) suitably selected. (Analogously as it was done in [2] relative to the rotation of the accompanying trihedron in azimuth).

Such a trihedron can be, for example, $O_1\xi\eta\zeta$ fixed in space. In this case the error equations are obtained from (3.16), (3.22), (3.24) and (3.25) if one assumes $\boldsymbol{\omega} = 0$. In place of (3.16) we obtain

$$\delta\xi^{\cdot\cdot} + \frac{g}{r}\,\delta\xi - \frac{3g}{r^2}\,\xi\,(|\mathbf{r} + \delta\mathbf{r}| - r) =$$
$$= \Delta n_\xi - 2\,(\Delta m_\eta \zeta^{\cdot} - \Delta m_\zeta \eta^{\cdot}) - \Delta m_\eta^{\cdot} \zeta + \Delta m_\zeta^{\cdot}\, \eta \qquad (\xi\eta\zeta) \qquad (4.1)$$

which follows also from (3.8), and in place of (3.22), (3.24) and (3.25) we get

$$\theta_\xi^{\cdot} = \Delta m_\xi, \qquad \delta\xi_2 = \theta_\eta\,\zeta - \theta_\zeta \eta, \qquad \delta\xi_1 = \delta\xi + \delta\xi_2 \qquad (\xi\eta\zeta) \qquad (4.2)$$

In Equations (4.1) and (4.2) Δn_ξ, Δn_η, Δn_ζ, Δm_ξ, Δm_η, Δm_ζ are respectively, the projections of $\Delta \mathbf{n}$ and $\Delta \mathbf{m}$ on the $\xi\eta\zeta$ axes . They can be obtained from Δn_x, Δn_y, Δn_z, Δm_x, Δm_y, Δm_z if $\alpha_{ij}(t)$ are known.

Equations (4.1) and (3.8) permit an interesting analogy. They are analogous to the variational equations of the motion of a particle of unit mass in the Earth's field of attraction when the motion of the point is perturbed by the forces appearing on the right-hand side of Equations (4.1) and (3.8). In particular, the equations (4.1) and (3.8) are analogous to the equations of motion of a particle in a satellite cabin [11]. The indicated analogy permits, in the latter case, the application to the analysis of the first group of error equations,for inertial navigation of the well developed methods of celestial mechanics.

If the attraction of celestial bodies other than the Earth is also considered, then the corresponding error equation will be analogous to the perturbed motion of a particle in the field of attraction of n bodies.

Equations (3.16), (3.22), (3.24) and (3.25) are the error equations for an arbitrary control system. They are valid, apparently, for the case when the orientation of the set $Oxyz$ is a given function of time, i.e. does not depend on the coordinates determined by the inertial system, as well as when the orientation of the trihedron is given as a function of coordinates determined by the inertial system. It is worth noting that in the latter case the position of the $Oxyz$ trihedron is perturbed by the errors of the coordinate determination.

The homogeneous equations of the first group (3.16) are exact. They describe the perturbed operating conditions of the inertial system not only for small but also for large perturbations.

Equations (3.22) and (3.24) of the second group are the equations for small deviations, since in passing from (3.20) to (3.22), the squares of $\delta\alpha_{ij}(t)$ were dropped. In order to obtain Equations (3.22) and (3.24) for large perturbations it is necessary to utilize the theory of finite and not small rotations [9] in transforming the exact equations (3.10) and (3.20).

Note that the second group of error equations for the inertial system is of the same form as was obtained in [2], and for given $\alpha_{ij}(t)$ can be integrated by quadratures, which follow from (4.2).

Let us obtain from Equations (3.16) the equations of perturbed operation for the particular inertial navigation systems investigated in [1 to 5].

Assuming in (3.16) $\omega_x = \omega_y = \omega_z = 0$

$$\delta x = \delta\xi_*', \quad \delta y = \delta\eta_*', \quad \delta z = \delta\zeta_*', \quad x = \xi_*, \quad y = \eta_* \quad z = \zeta_* \tag{4.3}$$

and noting that

$$r^2 = \xi_*^2 + \eta_*^2 + \zeta_*^2 \tag{4.4}$$

therefore, within the accuracy of terms up to second order of magnitude

$$|\mathbf{r} + \delta\mathbf{r}| - r = \xi_*\delta\xi_*' + \eta_*\delta\eta_*' + \xi_*\delta\zeta_*' \tag{4.5}$$

we obtain Equations (4.14) in the paper [2] for the inertial system where the integration takes place along the directions fixed in absolute space.

Directing the axis Oz of the set $Oxyz$ in the unperturbed state along $\mathbf{r}$, and noting that in this case

$$x = y = 0, \quad z = r \tag{4.6}$$

and within an accuracy of terms up to second order of magnitude

$$\delta x = r\beta, \quad \delta y = -r\alpha, \quad \delta z = \delta r, \quad |\mathbf{r} + \delta\mathbf{r}| - r = \delta z \tag{4.7}$$

we obtain Equations (3.9) in [2] for the system with three accelerometers located along the axes of the Darboux trihedron on the sphere surrounding the Earth.

The first two equations in (3.16) in this case become the small oscillation equations for the Schuler pendulum [5 and 12], two-gyroscope vertical [4], gyro horizon compass [3], and the system considered in [1] if in (4.5) one lets

$$\delta z = \delta r = 0 \tag{4.8}$$

In [1 to 5], the small oscillation equations for systems near the position of relative equilibrium are given. Equations (3.16) are exact. They immediately yield the equations for perturbed motion for arbitrary and not necessarily small deviations.

Let, for example, the trihedron $Oxyz$ in the unperturbed position be the Darboux trihedron on a sphere [1 to 5] surrounding the Earth. Let us denote it by $Ox_0y_0z_0$ retaining the notation xyz for the perturbed position. Let the perturbed position of the set xyz relative to the unperturbed be characterized by two angles * α and β according to the direction cosines table

$$\begin{array}{cccc} & x & y & z \\ x_0 & \cos\beta & 0 & \sin\beta \\ y_0 & \sin\alpha\sin\beta & \cos\alpha & -\sin\alpha\cos\beta \\ z_0 & -\cos\alpha\sin\beta & \sin\alpha & \cos\alpha\cos\beta \end{array} \tag{4.9}$$

The projections of the absolute angular velocity of the set $Oxyz$ upon its axes are expressed by means of the projections $\omega_{x_0}, \omega_{y_0}, \omega_{z_0}$ of the absolute angular velocity of the set $Ox_0y_0z_0$ on the axes $x_0y_0z_0$ and the angles α and β as follows:

$$\begin{aligned} \omega_x &= \omega_{x_0}\cos\beta + \omega_{y_0}\sin\alpha\sin\beta - \omega_{z_0}\cos\alpha\sin\beta + \alpha^{\cdot}\cos\beta \\ \omega_y &= \omega_{y_0}\cos\alpha + \omega_{z_0}\sin\alpha + \beta^{\cdot} \\ \omega_z &= \omega_{x_0}\sin\beta - \omega_{y_0}\sin\alpha\cos\beta + \omega_{z_0}\cos\alpha\cos\beta + \alpha^{\cdot}\sin\beta \end{aligned} \tag{4.10}$$

For the system with two accelerometers and the motion of the point O on a sphere of constant radius r it follows from (4.9) that

$$\delta x = -r\cos\alpha\sin\beta, \quad \delta y = r\sin\alpha, \quad \delta z = r(\cos\alpha\cos\beta - 1) \tag{4.11}$$

Substituting (4.10) and (4.11) into the first two equations in (3.16) we obtain after obvious groupings and simplifications **

$$\begin{aligned} &\beta^{\cdot\cdot} + \alpha^{\cdot 2}\sin\beta\cos\beta + 2\alpha^{\cdot}(\omega_{x_0}\sin\beta\cos\beta - \\ &- \omega_{y_0}\sin\alpha\cos^2\beta + \omega_{z_0}\cos\alpha\cos^2\beta) + \omega_{y_0}^{\cdot}(\cos\alpha - \cos\beta) + \\ &+ \omega_{z_0}^{\cdot}\sin\alpha + \omega_{x_0}^{\cdot}\sin\alpha\sin\beta + (\omega_0^2 - \omega_{z_0}^2\cos\alpha\cos\beta - \omega_{y_0}^2)\sin\beta\cos\alpha + \\ &+ \omega_{x_0}^2(\cos\beta - \cos\alpha)\sin\beta - \omega_{y_0}^2\sin^2\alpha\sin\beta\cos\beta + \\ &+ \omega_{x_0}\omega_{y_0}(\sin^2\beta - \cos^2\beta)\sin\alpha + \\ &+ \omega_{x_0}\omega_{z_0}(\cos^2\beta\cos\alpha - \cos\beta - \sin^2\beta\cos\alpha) + \\ &+ \omega_{y_0}\omega_{z_0}(2\cos\alpha\cos\beta - 1)\sin\alpha\sin\beta = 0 \end{aligned} \tag{4.12}$$

---

* The third rotation is unessential. It can be referred to the position of the $x_0y_0z_0$ set.

** The equations are homogeneous.

$$\alpha^{\cdot\cdot} \cos\beta - 2\alpha^{\cdot} \beta^{\cdot} \sin\beta + 2\beta^{\cdot} (-\omega_{x_0} \sin\beta + \omega_{y_0} \sin\alpha \cos\beta - \omega_{z_0} \cos\alpha \cos\beta) + \omega_{x_0}^{\cdot} (\cos\beta - \cos\alpha) + \omega_{y_0}^{\cdot} \sin\alpha \sin\beta - \omega_{z_0}^{\cdot} \cos\alpha \sin\beta + (\omega_0^2 - \omega_{x_0}^2 - \omega_{y_0}^2 (1 - \cos\alpha \cos\beta) - \omega_{z_0}^2 \cos\alpha \cos\beta) \sin\alpha - \omega_{x_0}\omega_{y_0} \cos\alpha \sin\beta - \omega_{x_0}\omega_{z_0} \sin\alpha \sin\beta + \omega_{y_0}\omega_{z_0} (\sin^2\alpha \cos\beta - \cos^2\alpha \cos\beta + \cos\alpha) = 0, \qquad \omega_0^2 = g / r \qquad (4.12) \text{ cont.}$$

Equations (4.10) coincide exactly with the equations of the Schuler pendulum [5 and 12] the suspension point of which moves on the sphere of radius r. Indeed, the pendulum equations, as projected on the axes xyz, are of the form

$$H_x^{\cdot} + \omega_y H_z - \omega_z H_y = - lF_y, \quad H_y^{\cdot} + \omega_z H_x - \omega_x H_z = lF_x \qquad (4.13)$$

For the Schuler pendulum [5 and 12], the projections of angular momentum are

$$H_x = mlr\omega_x, \qquad H_y = mlr\omega_y, \qquad H_z = 0 \qquad (4.14)$$

The components of $\mathbf{F}$ on the $x_0 y_0 z_0$ are

$$F_{x_0} = - mr (\omega_{y_0}^{\cdot} + \omega_{x_0}\omega_{z_0}), \qquad F_{y_0} = mr (\omega_{x_0}^{\cdot} - \omega_{y_0}\omega_{z_0})$$
$$F_{z_0} = mr (\omega_{x_0}^2 + \omega_{y_0}^2) - mg \qquad (4.15)$$

The substitution of (4.14), (4.15) and (4.10) into (4.13) gives immediately Equations (4.12). Equations (4.12) are also the equations of perturbed motion of systems [1, 3 and 4].

For constant $\omega_{x_0}, \omega_{y_0}, \omega_{z_0}$, Equations (4.12) possess a first integral. In order to obtain it, it is sufficient to multiply the first equation in (4.12) by $\beta^{\cdot}$, the second equation by $\alpha^{\cdot}\cos\beta$ and to add them. Integration of the sum yields

$$V = (\alpha^{\cdot} \cos\beta)^2 + \beta^{\cdot 2} - 2\omega_0^2 \cos\alpha \cos\beta + \omega_{z_0}^2 \cos^2\alpha \cos^2\beta + \omega_{x_0}^2 (\sin^2\beta + 2\cos\alpha\cos\beta) + \omega_{y_0}^2 (\sin^2\alpha \cos^2\beta + 2\cos\alpha\cos\beta) - 2\omega_{x_0}\omega_{y_0} \sin\alpha \sin\beta \cos\beta + 2\omega_{x_0}\omega_{z_0} (\cos\alpha \sin\beta \cos\beta - \sin\beta) + 2\omega_{y_0}\omega_{z_0} (\sin\alpha \cos\beta - \sin\alpha \cos\alpha \cos^2\beta) = \text{const} \qquad (4.16)$$

The Liapunov stability condition for the solution (4.12) follows from (4.16) as

$$\omega_0^2 - \omega_{x_0}^2 - \omega_{y_0}^2 - \omega_{z_0}^2 > 0 \qquad (4.17)$$

It was obtained earlier in [6 and 7] from consideration of the equations reducible to (4.12).

Condition (4.17) is a sufficient condition. The papers [6 and 13] show that condition (4.17) can be considered as a necessary stability condition if fully dissipative forces are assumed in the system. In this connection, it is necessary to note that dissipative forces in inertial systems lead to the occurence of velocity deviations, and the systems of inertial navigation in the absence of velocity correction, tend to be designed so as to avoid dissipation. Therefore, arbitrary introduction of dissipative forces into the system requires great care in the investigation of stability.

In order to obtain the equations for perturbed performance in a system with two accelerometers and large deviations for $r = r(t)$ when r is determined from auxiliary sources not connected with the operation of the inertial system, the r should be considered a given function of time in (4.11). If r is computed as a function of two coordinates determined by the inertial system, then in writing the equations of perturbed performance the variation of this function should be taken [2].

In order to obtain the equations for perturbed performance for large deviations of the system [2] with the variables α, β and δr, it is necessary in (3.16) to substitute

$$\delta x = -(r+\delta r)\cos\alpha\,\sin\beta, \qquad \delta y = (r+\delta r)\sin\alpha \qquad (4.18)$$
$$\delta z = (r+\delta r)\,(\cos\,\alpha\cos\beta - 1) + \delta r$$

in place of (4.11).

5. Let us investigate the stability of the inertial system performance for the case when the unperturbed Oz-axis of the set $Oxyz$ is directed along $\mathbf{r}$, and ω_x, ω_y, ω_z and r are constant. From (3.16)

$$|\mathbf{r}+\delta\mathbf{r}| - r = \delta z + o\,(\delta z) \qquad (5.1)$$

Retaining now on the right-hand side of (5.1) only δz, we note that Equations (3.16) become linear with constant coefficients, the characteristic equation of which is a cubic with respect to the square of the unknown $q = p^2$

$$q^3 + 2q^2(\omega_x^2+\omega_y^2+\omega_z^2) + q\,[-3\omega_0^4 + 3\omega_0^2(\omega_x^2 + \omega_y^2 - 2\omega_z^2) + (\omega_x^2+\omega_y^2+\omega_z^2)^2] - \omega_0^2(\omega_0^2-\omega_x^2-\omega_y^2-\omega_z^2)(2\omega_0^2+\omega_x^2+\omega_y^2-2\omega_z^2) = 0 \quad (5.2)$$

For stability (non asymptotic) Equation (5.2) must have, as is known, negative or zero roots, and to multiply roots of the characteristic equation of the system (3.16) should correspond linear, elementary divisors of the characteristic matrix.

In order to investigate the stability of the system (3.16), in the present case, one can utilize the fact that it can be considered as describing the motion of a particle of unit mass under the action of potential and gyroscopic forces.

The expression for the potential function can be written

$$U = -\tfrac{1}{2}\,[(\omega_0^2-\omega_y^2-\omega_z^2)\,\delta x^2 + (\omega_0^2-\omega_x^2-\omega_z^2)\,\delta y^2 - (2\omega_0^2+\omega_x^2+\omega_y^2)\,\delta z^2 + 2\omega_x\omega_y\delta x\delta y + 2\omega_x\omega_z\,\delta x\delta z + 2\omega_y\omega_z\,\delta y\delta z] \qquad (5.3)$$

The following expressions can be considered as gyroscopic forces:

$$2\omega_z\delta y^{\cdot} - 2\omega_y\,\delta z^{\cdot} \quad (xyz) \qquad (5.4)$$

since the matrix of coefficients of these forces is antisymmetric [14]. The system (3.16) has in this case the energy integral

$$\delta x^{\cdot 2} + \delta y^{\cdot 2} + \delta z^{\cdot 2} - 2U = \text{const} \tag{5.5}$$

which can be obtained directly if Equations (3.16) are added, multiplied respectively by $\delta x^{\cdot}$, $\delta y^{\cdot}$, $\delta z^{\cdot}$, and integrated

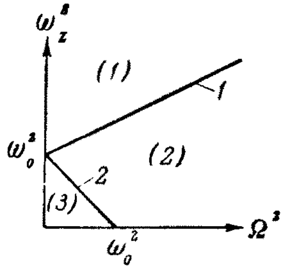

Fig. 1

If the gyroscopic forces (5.4) are rejected then there will remain only the potential forces. For stability of equilibrium under the action of only potential forces, the potential function must have a maximum at the equilibrium point. Since the potential function (5.3) is a quadratic form, the conditions for a maximum are the Sylvester conditions for positive-definiteness of of a quadratic form. In the present case they are the inequalities

$$\omega_0^2 - \omega_x^2 - \omega_y^2 - \omega_z^2 > 0, \qquad 2\omega_0^2 - 2\omega_z^2 + \omega_x^2 + \omega_y^2 < 0 \tag{5.6}$$

In Fig. 1 $\Omega^2 = \omega_x^2 + \omega_y^2$, the straight lines 1 and 2 corresponding to the equations $\omega_0^2 - \omega_z^2 - \Omega^2 = 0$ and $2\omega_0^2 + \Omega^2 - 2\omega_z^2 = 0$ are plotted. The figure shows that the regions defined by (5.6) do not intersect, and therefore, the potential function has no maximum.

Since in the present case the potential function is homogeneous of second degree then, according to the known theorem of Liapunov [15], the instability follows from the absence of the maximum without the necessity of considering the terms of higher orders.

Let us return to the gyroscopic forces (5.4).

In the regions (1) and (3) (see Fig. 1) where the degree of instability * of the conservative system is odd, the gyroscopic forces, according to the Thomson-Tait theorem [16], cannot stabilize the equilibrium.

In the region (2), where the degree of instability is even, the possibility of stabilization by gyroscopic forces remains in principle. This stabilization, as is known [15], has a temporary character and is destroyed by the forces of full dissipation.

The stabilization by gyroscopic forces results if, for example,

$$\omega_x^2 + \omega_y^2 = \omega_0^2, \qquad \omega_z^2 = \varepsilon^2 \tag{5.7}$$

where ε^2 is a sufficiently small quantity.

It can be easily shown that the polynomial (5.2) satisfies in this case the Hurwitz conditions. The discriminant Δ of the cubic equation obtained from (5.2) by substitution of the variable

$$y = q + {}^2/_3\ (\omega_0^2 + \varepsilon^2) \tag{5.8}$$

* The number of negative stability coefficients of Poincaré [15].

is negative

$$\Delta = -\omega_0^{10}\varepsilon^2 g^{-3} < 0 \tag{5.9}$$

All roots of the characteristic equation are therefore simple and purely imaginary.

6. In conclusion, let us investigate the stability of a system with three accelerometers when the orientation of the set $Oxyz$ is arbitrary, and $|\mathbf{r}|$ for $g(r)$ is given, in addition, from a source outside the inertial system.

In this case it is necessary to make the following assumption in the error equations (3.16)

$$|\mathbf{r} + \delta\mathbf{r}| - \mathbf{r} = 0 \tag{6.1}$$

and the error in $|\mathbf{r}|$ should be referred to the right-hand side parts in (3.16).

It follows from (3.16), (3.8) and (4.1) that the homogeneous error equations become

$$\delta\xi^{\cdot\cdot} + \frac{g}{r}\delta\xi = 0 \qquad (\xi\eta\zeta) \tag{6.2}$$

From (6.2) it follows that for r = const when $\omega_0^2 = g/r$ is constant, the perturbed motion of the system is stable for any $\omega_x(t)$, $\omega_y(t)$, $\omega_z(t)$.

In this case for given $\alpha_{ij}(t)$ the solution of (3.16) follows immediately from (6.2).

For constant ω_x, ω_y, ω_z the stability can be detected also without reference to equations (3.8) and (4.1).

In the present case the condition for the maximum of the potential function is reduced to one inequality

$$\omega_0^2 - \omega_x^2 - \omega_y^2 - \omega_z^2 > 0 \tag{6.3}$$

Outside of (6.3) the degree of instability is even and the equilibrium is stabilized by gyroscopic forces. The latter is easily proved by reviewing the characteristic equation which, if written in terms of the square of the unknown, is

$$q^3 + (3\omega_0^2 + 2\omega^2)\, q^2 + q\,(3\omega_0^4 + \omega^4) + \omega_0^2(\omega_0^2 - \omega^2)^2 = 0 \tag{6.4}$$

where for simplicity the notation

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2 \tag{6.5}$$

has been introduced.

The polynomial (6.4) satisfies the Hurwitz conditions, since always

$$(3\omega_0^2 + 2\omega^2)(3\omega_0^4 + \omega^4) - \omega_0^2(\omega_0^2 - \omega^2)^2 > 0 \tag{6.6}$$

The discriminant Δ of the cubic equation

$$y^3 + 3by + 2c = 0 \tag{6.7}$$

which is obtained from (6.4) by the change of variable

$$y = q + \frac{3\omega_0^2 + 2\omega^2}{3} \tag{6.8}$$

is nonpositive

$$\Delta = -\frac{4}{27}\omega_0^2\omega^6(4\omega_0^2 - \omega^2)^2 \leqslant 0 \tag{6.9}$$

If $\omega \neq 0$, $4\omega_0^2 - \omega^2 \neq 0$, and $\omega_0^2 - \omega^2 \neq 0$, then (6.4) has three different real roots. Consequently, the characteristic equation has three pairs of different purely imaginary roots.

For $\omega \neq 0$, $4\omega_0^2 - \omega_0^2 \neq 0$, and $\omega_0^2 = \omega^2$ Equation (6.4) has along with two real negative roots, also a zero root, and the characteristic equation has a multiple zero root.

If $\omega = 0$, then (6.4) has a triple root $q_{1,2,3} = -\omega_0^2$, and the characteristic equation, respectively, a pair of imaginary roots of the same multiplicity.

Finally, for $4\omega_0^2 - \omega^2 = 0$ Equation (6.4) has a multiple root $q_{2,3} = -\omega_0^2$, and the characteristic equation a pair of imaginary multiple roots.

It can be shown that when the roots of the characteristic equation are multiple, the elementary divisors of the characteristic matrix of the system (3.16) in the present case remain linear.

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